Mathematica for Function Tutorial in Module 8.2 (v. 7)

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Introduction to Computational Science: Modeling and Simulation for the Sciences
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Introduction

In this chapter, we deal with models that are driven by the data. In such a situation, we have data measurements and wish to obtain a function that roughly goes through a plot of the data points capturing the trend of the data, or **fitting the data**. Subsequently, we can use the function to find estimates at places where data does not exist or to perform further computations. Moreover, determination of an appropriate fitting function can sometimes deepen our understanding of the reasons for the pattern of the data.

In this module, we consider several important functions, some which we have already used. By being familiar with basic functions and function transformations, the modeler can sometimes more readily fit a function to the data.

Linear Function

y = mx + b

The concept of a linear function was essential in our discussions of the derivative and simulation techniques, such as Euler's Method. Here, we review some of the characteristics of functions whose graphs are lines.

The command in Quick Review Question 1 plots the graph of the linear function $y = 2t + 1$. This line has *y*-intercept 1, because $y = 1$ when $t = 0$. Thus, the graph crosses the *y*-axis when $t = 0$. With data measurements where *t* represents time, the *y*-intercept indicates the initial data value. The slope of this particular line is 2, which is the coefficient of *t*. Consequently, when we go over 1 unit to the right, the graph rises by 2 units.

Definitions A **linear function**, whose graph is a straight line, has the following form:

$$y = mx + b$$

The ***y*-intercept**, which is *b*, is the value of *y* when $x = 0$, or the place where the line crosses the *y*-axis. The **slope**, *m*, is the change in *y* over the change in *x*. Thus, if the line goes through points (x_1, y_1) and (x_2, y_2), the slope is as follows:

$$m = (y_2 - y_1)/((x_2 - x_1)$$

Quick Review Question 1

a. Execute the following *Mathematica* command, which plots the above function, *f*(*t*) = 2*t* + 1, from *t* = -3 to 3:

```
Plot[2 t + 1, {t, -3, 3}]
```

b. By replacing xxxxxx with the appropriate equation, complete the command below to graph *f* along with the equation of the line with the same slope as *f* but with *y*-intercept 3. The graph of *f* will appear in red, while the graph of the new function will be in blue.

```
Plot[{2 t + 1, xxxxxx }, {t, -3, 3},
 PlotStyle → {Red, Blue}
]
```

- **c. Copy the command from Part b, and change the second function to have a *y*-intercept of -3.**
- **d. Describe the effect that changing the *y*-intercept has on the graph of the line.**
- **e. Copy command from Part b and change the second function to have the same *y*-intercept as *f* but slope 3.**
- **f. Copy the command from Part b, and change the second function to have the same *y*-intercept as *f* but slope -3.**
- **g. Describe the effect that changing the slope has on the graph of the line.**

Quadratic Function

$y = a_2 t^2 + a_1 t + a_0$

In Module 2.3 on "Rate of Change" and Module 2.4 on "Fundamental Concepts of Integral Calculus," we considered a ball thrown upward off a bridge 11 m high with an initial velocity of 15 m/sec. The function of height of the ball with respect to time is the following quadratic function:

$$s(t) = -4.9t^2 + 15t + 11$$

The general form of a **quadratic function** is as follows:

$$f(x) = a_2 x^2 + a_1 x + a_0$$

where a_2, a_1, and a_0 are real numbers. The graph of the ball's height $s(t)$ in Figure 2.3.1 of the "Rate of Change" module is a **parabola** that is concave down. The next two Quick Review Questions develop some of the characteristics of quadratic functions.

Definition A **quadratic function** has the following form:

$$f(x) = a_2 x^2 + a_1 x + a_0$$

where a_2, a_1, and a_0 are real numbers. Its graph is a **parabola**.

- **Quick Review Question 2**

a. Execute the following *Mathematica* command, which plots the above function, $s(t) = -4.9t^2 + 15t + 11$, from $t = -1$ to 4:

```
Plot[-4.9 t^2 + 15 t + 11, {t, -1, 4}]
```

b. Give the *Mathematica* command to plot $s(t)$ in red and another function in blue with the same shape that crosses the y-axis at 2.

- **c. Using calculus, determine the time t at which the ball reaches its highest point. Verify your answer by referring to the graph.**
- **d. What effect does changing the sign of the coefficient of t^2 have on the graph?**
- **Quick Review Question 3 In this question, we consider various transformations on a function.**

  a. In *Mathematica*, plot t^2, $t^2 + 3$, and $t^2 - 3$ on the same graph in red, green, and blue, respectively.
- **b. Describe the effect of adding a positive number to a function.**
- **c. Describe the effect of subtracting a positive number from a function.**
- **d. In *Mathematica*, plot t^2, $(t + 3)^2$, and $(t - 3)^2$ on the same graph in red, green, and blue, respectively.**
- **e. Describe the effect of adding a positive number to the independent variable in a function.**
- **f. Describe the effect of subtracting a positive number from to the independent variable in a function.**
- **g. In *Mathematica*, plot t^2 and $-t^2$ on the same graph in different colors.**
- **h. Describe the effect of multiplying a function by -1.**
- **i. In *Mathematica*, plot t^2, $5t^2$, and $0.2\,t^2$ on the same graph in red, green, and blue, respectively.**
- **j. Describe the effect of multiplying the function by number greater than 1.**
- **k. Describe the effect of multiplying the function by positive number less than 1.**

Polynomial Function
$y = a_n t^n + \ldots + a_1 t + a_0$

Linear and quadratic functions are polynomial functions of degree 1 and 2, respectively. The general form of a **polynomial function of degree *n*** is as follows:

$$f(x) = a_n x^n + \ldots + a_1 x + a_0$$

where a_n, ..., a_1, and a_0 are real numbers and n is a nonnegative integer. The graph of such a function with degree greater than 1 consists of alternating hills and valleys. The quadratic function of degree 2 has one hill or valley. In general, a polynomial of degree n has at most n - 1 hills and valleys.

Definition A **polynomial function of degree *n*** has the following form:

$$f(x) = a_n x^n + ... + a_1 x + a_0$$

where a_n, ..., a_1, and a_0 are real numbers and n is a nonnegative integer.

■ Quick Review Question 4

a. Execute the following *Mathematica* command that plots the polynomial function $p(t) = t^3 - 4t^2 - t + 4$ from $t = -2$ to 5:

```
Plot[t^3 - 4 t^2 - t + 4, {t, -2, 5}]
```

■ **b. To what value does $p(t)$ go as t goes to infinity?**

■ **c. To what value does $p(t)$ go as t goes to minus infinity?**

■ **d. Give the *Mathematica* command to plot $p(t)$ in red and another function in blue with each coefficient having the opposite sign as in $p(t)$.**

■ **e. To what does the new function from Part d go as t goes to infinity?**

■ **f. To what does the new function from Part d go as t goes to minus infinity?**

Square Root Function

$y = \sqrt{x}$

The square root function, whose graph is below, is increasing and concave down. Its domain and range are the set of nonnegative real numbers.

```
Plot[Sqrt[t], {t, 0, 12}]
```

- **Quick Review Question 5 Write *Mathematica* commands to plot each of the following transformations of the square root function.**

  a. Move the graph to the right 5 units.
- **b. Move the graph up 3 units.**
- **c. Rotate the graph around the *x*-axis.**
- **d. Double the height of each point.**

Exponential Function
$y = y_0 e^{rt}$

In Module 3.2 on "Unconstrained Growth and Decay," we considered situations where the rate of change of a quantity, such as the size of a population, is directly proportional to the size of the population, such as $dP/dt = 0.1P$ with initial population $P_0 = 100$. As we saw, the solution to this differential equation is the exponential function $P = 100e^{0.1t}$ whose graph is in Figure 3.2.3 of that module. Similarly, the solution to the differential equation $dQ/dt = -0.000120968Q$ for radioactive decay is $Q = Q_0 e^{-0.000120968t}$ with graph in that module's Figure 3.2.4. As indicated in both examples, the coefficient is the initial amount and the coefficient of t is the continuous rate. For a positive rate, the function increases and is concave up; while a negative rate results in a decreasing, concave-up function.

The base can be any positive real number, not just e, which is approximately 2.71828. For example, we can express $P = 100e^{0.1t}$ as an exponential function with base 2. Setting $100(2^{rt})$ equal to $100e^{0.1t}$, we cancel the 100s and take the natural logarithm of both sides.

$100e^{0.1t} = 100(2^{rt})$

$e^{0.1t} = 2^{rt}$

$0.1t = \ln(2^{rt})$

$0.1t = rt \ln(2)$

$r = 0.1/\ln(2) = 0.14427$

Thus, $P = 100e^{0.1t} = 100(2^{0.14427t})$.

Definition An **exponential function** has the following general form:

$P = P_0 a^{rt}$

where P_0, a, and r are real numbers.

Quick Review Question 6

a. Define in *Mathematica* an exponential function $u(t)$ with initial value 500 and continuous rate 12%.

- **b. Plot this function in *Mathematica*.**
- **c. On the same graph, plot exponential functions with initial value 500 and continuous rates of 12%, 13%, and 14%. Which rises the fastest?**
- **d. Express the function $u(t)$ as an exponential function with base 4.**
- **Quick Review Question 7**

a Define in *Mathematica* an exponential function $v(t)$ with initial value 5 and continuous rate -82%.

- **b. Plot this function in *Mathematica*.**
- **c. Plot $v(t)$ in red and $v(t)$ + 7 in blue on the same graph.**
- **d. What effect does adding 7 have on the graph?**
- **e. As t goes to infinity, what does $v(t)$ approach?**
- **f. As t goes to infinity, what does $v(t)$ + 7 approach?**
- **g. Copy the answer to Part b to another cell. In the copy, plot $v(t)$ and -$v(t)$.**
- **h. What effect does negation (multiplying by -1) have on the graph?**
- **i. Copy the answer to Part g to another cell. In the copy, plot $v(t)$ and 7 - $v(t)$.**
- **j. As t goes to infinity, what does 7 - $v(t)$ approach?**
- **k. Give the value of 7 - $v(t)$ when t = 0.**
- **Quick Review Question 8 This question considers a function that has an independent variable t as a factor and as an exponent.**

a. Plot $12te^{-2t}$ from $t = 0$ to $t = 5$.

- **b. Initially, with values of t close to 0, give the factor that has the most impact, t or e^{-2t}.**
- **c. As t gets larger, give the factor that has the most impact, t or e^{-2t}.**

Logarithmic Function

$y = \log_b(x)$, $x > 0$

$y = \log(x)$, $x > 0$
$y = \ln(x)$, $x > 0$

In Module 3.2 on "Unconstrained Growth and Decay," we employed the logarithmic function to obtain an analytical solution to the differential equation $dP/dt = 0.1P$ with initial population $P_0 = 100$. In that same module, the logarithmic function was useful in solving a problem to estimate the age of a mummy.

John Napier, a Scottish baron who considered mathematics a hobby, published his invention of logarithms in 1614. Unlike most other scientific achievements, his work was not built on that of others. His highly original invention was welcomed enthusiastically. It was found that by using logarithms, problems of multiplication and division could be reduced to much simpler problems of addition and subtraction.

By definition, m is the **logarithm to the base 10**, or the **common logarithm**, **of** n written as $\log_{10} n = m$ or $\log n = m$, provided m is the exponent of 10 such that 10^m is n or

$$\log_{10} n = m \quad \textbf{if and only if } n = 10^m$$

A logarithm is an exponent, in this case, an exponent of 10. Thus,

$\log_{10} 1000 = 3$ because $1000 = 10^3$

$\log_{10} 1{,}000{,}000 = 6$ because $1{,}000{,}000 = 10^6$

$\log_{10} 0.01 = -2$ because $0.01 = 10^{-2}$

Because 10^m is always positive, we can only take the logarithm of positive numbers, so that the domain of a logarithmic function is the set of positive real numbers. However, the exponent m, which is the logarithm, can take on values that are positive, negative, or zero. Thus, the range of a logarithmic function is the set of all real numbers. Below is the graph of the common logarithm. Because the logarithm is an exponent, the logarithmic function increases very slowly, and the graph is concave down.

```
Plot[Log[x], {x, 0, 10}]
```

In scientific applications, we frequently employ the **logarithm to the base *e*** or the **natural logarithm**. The notation for $\log_e n$ is $\ln n$. Similarly to the common logarithm, we have the following equivalence:

$$\ln n = m \quad \textbf{if and only if } n = e^m$$

Moreover, the graph of the natural logarithm has a similar shape to that of the common logarithm above.

In *Mathematica*, the common logarithm of n is ***Log*[10, *n*]**. Thus, the following call to the function returns 3:

```
Log[10, 1000]
```

The natural logarithm of n is ***Log*[*n*]** in *Mathematica*. Thus, *Log*[50.0] returns 3.91202 because $e^{3.91202}$ is 50.0. Bases other than e or 10 are permissible as long as the base is greater than 1. In general, *Mathematica*'s ***Log*[*b*, *n*]** is $\log_b n$.

Definitions The **logarithm to the base *b* of *n***, written $\log_b n$, is m if and only if b^m is n. That is, $\log_b n = m$ is equivalent to $n = b^m$. The **common logarithm** of n, usually written **log *n***, has base 10; and the **natural logarithm** of n, usually written **ln *n***, has base e.

In comparing the graph of ln x to that of x and $\sqrt{x}$ below we see that the linear and square root functions dominate the logarithmic function, which is in blue.

```
Plot[{x, √x, Log[x]}, {x, 0, 10},
 PlotStyle → {Red, Green, Blue}]
```

- **Quick Review Question 9**

a. Evaluate $\log_2 8$.

- **b. Write $y = \log 7$ as a corresponding equation involving an exponential function.**
- **c. Evaluate ln ($e^{5.3}$).**
- **d. Evaluate $10^{\log(6.1)}$.**

Logistic Function

In Module 3.3 on "Constrained Growth," we modeled the rate of change of a population with a carrying capacity that limited its size. The model incorporated the following differential equation with carrying capacity M and initial population P_0:

$$\frac{dP}{dt} = r(1 - \frac{P}{M})P$$

The resulting analytical solution, which is a logistic function, is as follows:

$$P = \frac{MP_0}{(M - P_0)e^{-rt} + P_0}$$

Figure 3.3.1 of the "Constrained Growth" module depicts the characteristic S-curve of this function.

- **Quick Review Question 10**

a. Plot the logistic function with initial population P_0 = 20, carrying capacity M = 1000,

and instantaneous rate of change of births r = 50% = 0.5 from t = 0 to 16 to obtain a graph as in Figure 3.3.1 of Module 3.3 on "Constrained Growth."

- **b. On the same graph in red, green, and blue, respectively, plot three logistic functions that each have M =1000 and r = 0.5 but P_0 values of 20, 100, and 200.**
- **c. What effect does P_0 have on a logistic graph?**
- **d. On the same graph, plot three logistic functions that each have M =1000 and P_0 = 20 but r values of 0.2, 0.5, and 0.8.**
- **e. What effect does r have on a logistic graph?**
- **f. On the same graph in red, green, and blue, respectively, plot three logistic functions that each have P_0 = 20 and r = 0.5 but M values of 1000, 1300, and 2000.**
- **g. What effect does M have on a logistic graph?**

Trigonometric Functions

The sine and cosine functions are employed in many models where oscillations are involved. For example, projects in Module 6.4 on the "Predator-Prey Model" considered seasonal birth rates and fishing and employed the cosine and sine functions, respectively, to achieve periodicity.

To define the trigonometric functions sine, cosine, and tangent, we consider the point (x, y) on the unit circle (see Figure 8.2.6 in text). For the angle t off the positive x-axis, with t being positive in the counterclockwise direction and negative in the clockwise direction, the definitions of these trigonometric functions are as follows:

sin $t = y$

cos $t = x$

tan $t = y / x$

For example, if x = 0.6 and y = 0.8, then t is approximately 0.927295 radians, so that the following hold:

sin(0.927295) = 0.8

cos(0.927295) = 0.6

tan(0.927295) = 0.8 / 0.6 ≈ 1.33

For an angle of 0 radians, the opposite side, y, is zero, so that sin(0) = 0. An angle of $\pi/2$ results in (0, 1) being the point on the unit circle and the sine function achieving its maximum value of 1. The sine returns to 0 for the angle π = 180º. Then, sin(t) obtains its minimum, namely –1, at $3\pi/2$, where the point on the unit circle point is (0, -1). At $t = 2\pi$ = 360º, the sine function starts cycling through the same values again. The following commands generate one cycle of the sine and cosine functions, respectively.

```
Plot[Sin[t], {t, 0, 2 π}]
```

```
Plot[Cos[t], {t, 0, 2 π}]
```

- **Quick Review Question 11**

  a. Evaluate sin *t* where $x = 0.6$ and $y = 0.8$.

- **b. Evaluate sin(π/3) where the corresponding point on the unit circle is $(\frac{1}{2}, \frac{\sqrt{3}}{2})$.**
- **c. Give the domain of the sine function.**
- **d. Give the range of the sine function.**
- **e. Give the sine's period, or length of time before the function starts repeating.**
- **f. Is sin *t* positive or negative for values of *t* in the first quadrant?**
- **g. Is sin *t* positive or negative for values of *t* in the second quadrant?**
- **h. Is sin *t* positive or negative for values of *t* in the third quadrant?**
- **i. Is sin *t* positive or negative for values of *t* in the fourth quadrant?**
- **Quick Review Question 12**

  a. Evaluate cos(0).

- **b. Evaluate cos(π/2).**
- **c. Evaluate cos(π).**
- **d. Evaluate cos(3π/2).**
- **e. Evaluate cos(π/3) where the corresponding point on the unit circle is $(\frac{1}{2}, \frac{\sqrt{3}}{2})$.**
- **f. Give the maximum value of cos *t*.**
- **g. Give the minimum value of cos *t*.**
- **h. Give the domain of the cosine function.**
- **i. Give the period of the cosine function.**
- **j. Is cos *t* positive or negative for values of *t* in the first quadrant?**
- **k. Is cos *t* positive or negative for values of *t* in the second quadrant?**
- **l. Is cos *t* positive or negative for values of *t* in the third quadrant?**
- **m. Is cos *t* positive or negative for values of *t* in the fourth quadrant?**

For a function of the form $f(t) = A \sin(Bt)$ or $g(t) = A \cos(Bt)$, where A and B are positive numbers, A is the **amplitude**, or maximum value of the function from the horizontal line going through the middle of the function. For example, $h(t) = 2 \sin(7t)$ has amplitude 2; the function oscillates between y values of –2 and 2. Because the period of the sine and cosine functions is 2π, the period of f and g above is $2\pi/B$. When $t = 0$, $Bt = 0$. When $t = 2\pi/B$, $Bt = B(2\pi/B) = 2\pi$. Thus, the period of $h(t) = 2 \sin(7t)$ is $2\pi/7$.

- **Quick Review Question 13 Plot each pair of functions on the same graph with the first function**

in red and the second in blue. Recall that in *Mathematica*, sin *t* is *Sin*[*t*], and cos *t* is *Cos*[*t*].

a. sin *t* and 2 sin(7*t*).

- **b. sin *t* and a function involving sine that has amplitude 5 and period 6π.**
- **c. sin *t* and a function involving sine that has minimum value –2 and maximum value 4.**
- **d. sin *t* and a function involving sine that has amplitude 4 and crosses the *t*-axis at each of the following values of *t:* …, -π/6, π/3, 5π/6, ….**
- **e. cos *t* and a function involving cosine that has amplitude 3, period π, achieves its maximum value at *t* = π/5, and the maximum value is 2.**
- **f. sin(5*t*) and e^{-t}sin(5*t*). The latter is a function of decaying oscillations. The general form of such a function is $Ae^{-Ct}\sin(Bt)$, where *A*, *B*, and *C* are constants.**

The tangent function is also periodic. Because tan *t* = *y* / *x*, for a corresponding point (x, y) on the unit circle (see Figure 8.2.6), tan *t* = sin *t* / cos *t*. The command below generates a graph of this function, and the next Quick Review Question explores some of its properties.

```
Plot[Tan[t], {t, -π, π}]
```

■ Quick Review Question 14

a. Evaluate tan(π/3) where the corresponding point on the unit circle is ($\frac{1}{2}$, $\frac{\sqrt{3}}{2}$).

■ b. Evaluate tan(0).

■ c. Evaluate tan(π).

■ d. Evaluate tan(π/2).

■ e. As *t* approaches π/2 from values less than π/2, what does tan *t* approach?

■ f. As *t* approaches π/2 from values greater than π/2, what does tan *t* approach?

■ g. Evaluate tan(-π/2).

■ h. As *t* approaches -π/2 from values greater than -π/2, what does tan *t* approach?

■ i. As *t* approaches -π/2 from values less than -π/2, what does tan *t* approach?

■ j. Give the range of the tangent function.

■ k. Give all the values between –2π and 2π for which tan *t* is not defined.

■ l. Give an angle in the third quadrant that has the same value of tan *t*, where *t* is in the first quadrant.

■ m. Give an angle in the fourth quadrant that has the same value of tan *t*, where *t* is in the second quadrant.

■ n. Give the period of the tangent function.